Black Hole Geometries in Noncommutative String Theory

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Abstract: We obtain a generalized Schwarzschild (GS-) and a generalized Reissner-Nordstrom (GRN-) black hole geometries in $(3+1)$-dimensions, in a noncommutative string theory. In particular, we consider an effective theory of gravity on a curved D_3-brane in presence of an electromagnetic (EM-) field. Two different length scales, inherent in its noncommutative counter-part, are exploited to obtain a theory of effective gravity coupled to an $U(1)$ noncommutative gauge theory to all orders in Θ. It is shown that the GRN-black hole geometry, in the Planckian regime, reduces to the GS-black hole. However in the classical regime it may be seen to govern both Reissner-Nordstrom and Schwarzschild geometries independently. The emerging notion of $2D$ black holes evident in the frame-work are analyzed. It is argued that the D-string in the theory may be described by the near horizon $2D$ black hole geometry, in the gravity decoupling limit. Finally, our analysis explains the nature of the effective force derived from the nonlinear EM-field and accounts for the Hawking radiation phenomenon in the formalism.

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1 Introduction

Black holes are known as the classical solutions to the Einstein's general theory of relativity (GTR). However, the loss of information from a black hole is not governed by a pure classical physics. At least, it requires a semi-classical phenomenon, where the space-time is treated classically in presence of a quantized matter field in the theory.

Very recently, the information loss problem was re-visited by Hawking using a path integral formalism [1]. Since such a process involves a decay of pure quantum states into mixed ones, it gives rise to a non-unitary quantum theory gravity. In fact, the idea dates back to the seminal work [2] on Hawking radiation from a black hole. It was established that black holes radiate quantum mechanically. The radiation is not exactly thermal in nature, rather it arises due to the particle creation at the event horizon of a black hole by an external source for the matter field [2]-[5]. However there is a caveat in the formalism, as the strong curvature of space-time at Planck scale is usually ignored to estimate the Hawking radiation.

In the context a D-brane [6] may be seen to provide a natural frame-work to address the problem of Hawking radiation from a black hole. The idea of strong space-time curvature on the D_3-brane world-volume is supported by the fact that they are the Planck scale probes in a string theory. Interestingly, some of the issues in quantum gravity may be addressed by developing an appropriate formalism on a curved D-brane. Among the recent developments, the nonlinear electrodynamics on a D_3-brane [7]-[9] appears to be a potential candidate to address some of the semi-classical solutions, such as shock wave and black hole geometries. Above all, several aspects of quantum gravity have been addressed on a D-brane world-volume in the recent literatures [10]-[18].

In particular, the noncommutative D-brane world-volume [19] may be seen to be one such relevant frame-work, to address the Hawking radiation phenomenon and the information loss paradox. In fact, the closed string decoupling limit in a noncommutative open string theory [20] turns out to be a powerful tool as the gravity decouples completely from the theory, but at the Planck scale. The cancellation of strong intrinsic curvature in the regime by the strong nonlinearity in the Maxwell theory is remarkable. In fact, the extrinsic curvature due to the noncommutative gauge field in the theory dominates over that of the gravity. As a consequence the frame-work allows one to formulate an effective quantum theory of gravity without any ambiguity with the principle of equivalence. In particular a D_3-brane world-volume incorporating the Einstein's GTR coupled to the nonlinear Maxwell's theory may yield a plausible frame-work at Planck scale. Interestingly, some of the recent field theoretic models [21, 22], similar to that of a string, have been worked out in the literatures. However the stringy frame-work being natural at Planck scale, provides a wide perspective.

In this paper, we begin with a D_3-brane in presence of an EM-field in a open bosonic string theory.[3] We consider a static gauge condition to incorporate a nontrivial induced metric on the brane world-volume. The brane dynamics is worked out for both ordinary as well as its noncommutative counter-part in the theory. We obtain a GS- and a GRN-black holes on the D_3-brane in the effective theory. The mass and the charge of the black holes are shown to receive noncommutative corrections to all order in Θ. The GRN-black hole geometry is analyzed for various values of its effective parameters in the classical and subsequently in the Planckian regimes. It is shown that the GRN-geometry reduces to a RN-like geometries in the semi-classical regime. At the other extreme, the GRN geometry is shown to describe a GS-black hole. In addition, the noncommutative frame-work gives rise to the notion of a two dimensional black hole in the theory. The Hawking temperature is obtained for the black holes to analyze the quantum radiation phenomenon. Hawking radiation from the black holes is explained with the help of the nonlinear **E**-field in the theory.

We plan to organize the paper as follows. In section 2, we describe the set-up to generalize a flat D_3-brane to a curved one. The effective description leading to noncommutative scaling in the frame-work is given in section 3. In section 4, we perform a series of orthogonal rotations to establish a scaled "spherical polar"coordinate system. The generalized black hole geometries are obtained and analyzed in section 5. Exact $2D$ near horizon black holes are obtained in section 6. Finally, we conclude with some remarks in section 7.

2 D_3-brane geometry: Set-up

Dp-branes are $(p+1)$-dimensional, short distance, probe in string theory. The open bosonic string boundary fluctuations make them dynamical. The closed string background fields, the metric $\bar{g}_{MN}(X)$ and the simplectic two-form $\bar{\mathcal{B}}_{MN}(X)$ in the theory, respectively, give rise to the induced field $g_{\mu\nu}(Y) = \partial_\mu X^M \partial_\nu X^N \bar{g}_{MN}$ and $\mathcal{B}_{\mu\nu}(Y) = \partial_\mu X^M \partial_\nu X^N \bar{\mathcal{B}}_{MN}$ on the brane, where $(M, N = 1, 2, \ldots 26)$. The $U(1)$ gauge potential $\mathcal{A}_M(X)$ at the open string boundary, substantially contribute $A_\mu(Y)$ on the D_3-brane, which may be seen due to the 22-Dirichlet conditions there. Thus all the fields, on the brane world-volume, possess a nontrivial space-time dependence [23], which gives rise to the strong curvature in the theory. In addition, the D_3-brane are charged under the Ramond-Ramond forms in type II string theories and may be viewed as nonperturabative extended objects [6]. These facts together may imply that a curved

[3] The mass-shell condition has been worked out, by one of us in a collaboration [11], to show the natural emergence of two different length scales in theory. Recently, we have exploited the large and small dimensions in a noncommutative string theory [14] to investigate a scattering phenomenon on a D_3-brane. It was shown that the S-matrix in the theory generalizes that of a point particle scattering in Einstein's theory at Planck scale [24]. It was argued that the physical consistency in quantum gravity imposes a Lorentzian signature on the emerging D-string in the theory. In fact, the underlying idea of two scales in a noncommutative string theory is precisely in agreement with the scaling arguments of Verlinde and Verlinde [25, 26].

world-volume theory provides a viable frame-work to address some of the issues in quantum gravity.

On the other hand, a construction for the D-brane dynamics in a open bosonic string theory has been worked out for a constant background two form $\bar{\mathcal{B}}_{MN}$ field. In addition, the induced metric $g_{\mu\nu}$ has been treated as a constant, for simplicity, in the theory. Then, the brane world-volume is described by the zero modes of the induced fields. It may appropriately be viewed as the flat D_3-brane in the asymptotic limit, *i.e.* $r \to \infty$, of a more general curved brane. In fact, the closed string modes are tangential to the D_3-brane, which means that string bulk dynamics is not completely decoupled in the theory. In addition, in a static gauge condition on the space-time, the nontrivial metric $\tilde{g}_{\mu\nu} = g_{\mu\nu}$. Given the facts, it may be possible construct a curved D_3-brane dynamics by incorporating the non-zero modes of the induced metric in the theory. Schematically, the space-time curvature as develops on the brane may be represented by a cigar diagram in figure 1.

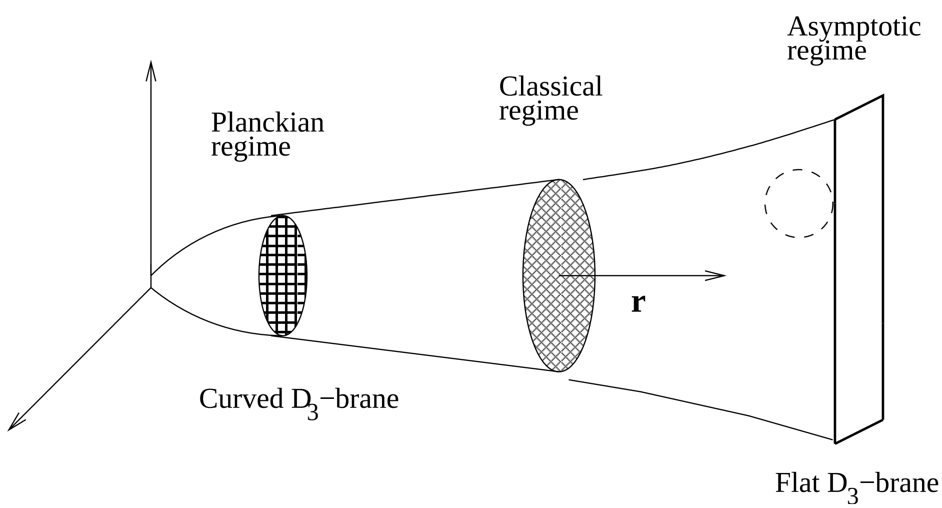

Figure 1: The space-time curvature on the D_3-brane in classical and Planckian regimes. The curvature begins to pierce into the D_3-brane world-volume in the classical regime itself and gradually develops its way to the Planckian regime.

2.1 Asymptotic description

We begin with a flat D_3-brane, in presence of a constant two form $\mathcal{B}$ induced on its world-volume (y_1, y_2, y_3, y_4). We consider an Euclidean world-volume signature $(+,+,+,+)$ all through the paper. The Minkowski signature may be obtained by an analytic continuation $y_4 \to it$. The uniform EM-field on the D_3-brane may be expressed by its arbitrary components $\mathbf{E} = (0, E_2, E_3)$ and $\mathbf{B} = (0, B_2, B_3)$. Since the induced metric $g_{\mu\nu}$ and the two form $\mathcal{B}_{\mu\nu}$ are constants on the brane world-volume, the DBI-action describes a nonlinear electrodynamics

and may be governed by the asymptotic limit of a string theory. The brane dynamics is known to be governed by the Dirac-Born-Infeld (DBI) action. It is given by

$$\begin{aligned} S_{\rm DBI} &= T_D \int d^4y \left(\sqrt{g} - \sqrt{(g+\bar{\mathcal{B}})} \right) \\ &= -\frac{1}{4g_{\rm nl}^2} \int d^4y \ \sqrt{g}\ g^{\mu\nu} g^{\lambda\rho}\ \mathcal{B}_{\mu\lambda}\mathcal{B}_{\nu\rho} \ , \end{aligned} \tag{1}$$

where T_D denotes the D_3-brane tension, $g \equiv \det g_{\mu\nu}$, $(g+\bar{\mathcal{B}}) \equiv \det(g+\mathcal{B})_{\mu\nu}$ and $g_{\rm nl}$ is a coupling constant in the nonlinear Maxwell theory. Though the metric $g_{\mu\nu} = g\delta_{\mu\nu}$ and the two-form $\mathcal{B}_{\mu\nu}$ are constants, its counter-part on the noncommutative D_3-brane may precisely be given by a nontrivial effective metric [19]

$$G_{\mu\nu} = g_{\mu\nu} - \left(\mathcal{B}g^{-1}\mathcal{B}\right)_{\mu\nu} \ . \tag{2}$$

Explicitly the effective metric is worked out in, natural units ($\hbar = 1 = c$), and may be expressed as

$$ds^2 = \left(g - G_N^2\ g^{-1}\mathbf{E}^2\right)\left[dy_3^2 + dy_4^2\right] + \left(g + G_N^2\ g^{-1}\mathbf{B}^2\right)\left[dy_1^2 + dy_2^2\right] \ , \tag{3}$$

where G_N denotes the Newton's constant. The notations used are $E_i = i\mathcal{B}_{i4}$ for $i = (1,2,3)$ and $B_i = \frac{1}{2}\epsilon_{ijk}\mathcal{B}_{jk}$ is defined with a normalization $\epsilon_{123} = 1$. In particular, in a typical cartesian coordinate system $\mathcal{J}_C$, *i.e.* (τ, x, y, z), the metric may be re-expressed as

$$ds^2 = \left(g - G_N^2\ g^{-1}\mathbf{E}^2\right)\left[d\tau^2 + dz^2\right] + \left(g + G_N^2\ g^{-1}\mathbf{B}^2\right)\left[dx^2 + dy^2\right] \ , \tag{4}$$

The EM-field may be seen to be governed by an (anti-) parallel configuration. In the spherical polar coordinate ($\mathcal{J}_S$-) system (τ, r, θ, ϕ), the effective metric becomes

$$ds^2 = \left(g - G_N^2\ g^{-1}\mathbf{E}^2\right)\left[d\tau^2 + dr^2\right] + \left(g + G_N^2\ g^{-1}\mathbf{B}^2\right) r^2 d\Omega^2 \ . \tag{5}$$

The geometry describes a spherically symmetric spike solution in nonlinear electrodynamics [8]. The transformation between $\mathcal{J}_O \to \mathcal{J}_S$-coordinates, may also be viewed in two steps. First the $\mathcal{J}_S$-coordinates are obtained from that in $\mathcal{J}_O$-system by the map

$$\begin{aligned} &\tau \to \tau \ , \qquad z = r\cos\theta \ , \\ &x = r\sin\theta\cos\phi \ \ \text{and} \ \ \ y = r\cos\theta\sin\phi \ . \end{aligned} \tag{6}$$

In the second step, an orthogonal rotation $\mathcal{R}_\theta$ of (r,θ)-plane in $\mathcal{J}_S$-system, by an arbitrary angle θ, around the symmetry axes (τ, ϕ) leads to the effective metric (3).

Performing two successive orthogonal rotations in anti-clock-wise directions, respectively, around $\hat{\phi}$ axis by an angle $\pi/2$ and around $\hat{r}$ by π in $\mathcal{J}_S$-system, one finds

$$\hat{\tau} \to \hat{\tau} \ , \qquad \hat{r} \to \hat{\theta} \ , \qquad \hat{\theta} \to \hat{r} \qquad \text{and} \quad \hat{\phi} \to -\hat{\phi} \ . \tag{7}$$

In the new coordinate system, the effective metric (3) becomes

$$ds^2 = \left(g - G_N^2\, g^{-1}\mathbf{E}^2\right) d\tau^2 + \left(g + G_N^2\, g^{-1}\mathbf{E}^2\right) dr^2$$
$$+ \left(g - G_N^2\, g^{-1}\mathbf{B}^2\right) r^2 d\theta^2 + \left(g + G_N^2\, g^{-1}\mathbf{B}^2\right) r^2 \sin^2\theta\, d\phi^2\ . \quad (8)$$

The orthogonal transformations, between different coordinate systems, show that different geometries (4), (5) and (8) may directly be obtained from the eq.(3) by an appropriate choice of orthogonal axes in the theory. For instance, the EM-field breaks the spherical symmetry $S_2 \rightarrow S_2^{\Theta}$ in the new coordinates (8), which is otherwise preserved in (5). In other words, the noncommutative geometry seems to be generated in the new coordinates. We shall see that the effective metric components (8) receive all order Θ corrections in the theory.

2.2 Classical regime

The asymptotic description (1) may be generalized to include a slow variation in the metric $g_{\mu\nu}$ in the classical regime. Then the D_3-brane dynamics would be governed by coupling Einstein's theory to an appropriate DBI-action. However, we consider a static gauge condition on space-time to begin with. Then the bulk metric may be viewed on the world-volume and the complete action becomes

$$S = \frac{1}{16\pi G_N} \int d^4y\ \sqrt{g}\ R\ +\ S_{\rm DBI}\ , \quad (9)$$

where R is the scalar curvature in the theory. The expansion under the square-root in the action (1), in the regime, is worked out. Then the action (9) becomes

$$S = \int d^4y\ \sqrt{g}\ \left(\frac{1}{16\pi G_N}\ R\ -\frac{1}{4} g^{\mu\nu} g^{\lambda\rho}\ \mathcal{F}_{\mu\lambda}\mathcal{F}_{\nu\rho}\ +\ \mathcal{O}(\mathcal{F}^4)\ +\ \ldots \right)\ . \quad (10)$$

The $U(1)$ gauge invariant field strength in the theory is governed by $\bar{\mathcal{F}}_{\mu\nu} = (\mathcal{B}+2\pi\alpha'\bar{F})_{\mu\nu}$. The higher order terms in gauge field may be ignored in the classical regime. Then, the nonlinear action (10) resembles to the Einstein's theory coupled to the Maxwell's. The equation of motion for the gauge field becomes

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0\ . \quad (11)$$

In addition, the metric variation in (10) leads to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = (8\pi G_N)\, T_{\mu\nu}\ , \quad (12)$$

where the energy momentum tensor is given by

$$T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta S_{\rm DBI}}{\delta g^{\mu\nu}}\ ,$$
$$= \frac{1}{2}\left(\frac{1}{4} g_{\mu\nu} \mathcal{F}_{\mu'\nu'} \mathcal{F}^{\mu'\nu'} - \mathcal{F}_{\mu\lambda}\mathcal{F}^{\lambda}_{\nu} \right)\ . \quad (13)$$

The gauge potential $A_\mu = (A_\tau,\ A_r,\ A_\theta,\ A_\phi)$ is governed by the equations of motion (11) and is given by

$$A_\mu = \left(\frac{-iQ_e}{r},\ 0,\ 0,\ Q_m \cos\theta \right) \ , \tag{14}$$

where Q_e and Q_m are constants, respectively, denote the electric and magnetic charges in the theory. Since the components of the gauge potential (14) remain unchanged in their magnitude under the rotations (7), the EM-field is worked out in the transformed coordinate frame. Then the EM-field may be seen to be governed by an anti-parallel configuration

$$\mathbf{E} = -\frac{Q_e}{r^2}\ \hat{r} \qquad \text{and} \qquad \mathbf{B} = \frac{Q_m}{r^2}\ \hat{r}\ . \tag{15}$$

To leading order in nonlinearity, *i.e.* for $\mathcal{B} = 0$, the theory (10) precisely reduces to the Einstein's theory coupled to the Maxwell's. In presence of both the charges, the geometry may be seen to be governed by

$$\begin{aligned} ds^2 \ &= \ \left(1 - \frac{2G_N M}{r} + \frac{G_N Q_e^2}{r^2}\right) d\tau^2 + \left(1 - \frac{2G_N M}{r} + \frac{G_N Q_e^2}{r^2}\right)^{-1} dr^2 \\ &\quad + \left(1 - \ \frac{G_N Q_m^2}{r^2}\right) r^2 d\theta^2 \ + \ \left(1 - \ \frac{G_N Q_m^2}{r^2}\right)^{-1} r^2 \sin^2\theta \ d\phi^2 \ . \end{aligned} \tag{16}$$

For $Q_m = 0$, the solution restores the spherical symmetry and governs the Reissner-Nordstrom (RN-) black hole geometry in the regime. Interestingly, some generalizations of RN-geometry, in the classical regime, have been studied [21, 22] in a different context.

On the other hand, in the classical regime $F_{\mu\nu} \neq 0$. Then, the expression for the effective metric (2) becomes

$$G_{\mu\nu} = g_{\mu\nu} - \left(\bar{\mathcal{F}} g^{-1} \bar{\mathcal{F}}\right)_{\mu\nu} \ + \ \mathcal{O}(\mathcal{F}^4) \ + \ \ldots \ \ . \tag{17}$$

Since $T_{\mu\nu}$ is weak, one may approximate the gravitational solution in the theory by the Schwarzschild black hole geometry. A semi-classical solution to the equations of motion (11) and (12) is worked out in the classical regime. Ignoring the higher order terms in the modified metric (17), we obtain

$$\begin{aligned} ds^2 \ &= \ \left(1 - \frac{2G_N M}{r}\right) \left(1 - \frac{G_N^2 Q_e^2}{r^4}\right) d\tau^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} \left(1 - \frac{G_N^2 Q_e^2}{r^4}\right)^{-1} dr^2 \\ &\quad + \left(1 - \ \frac{G_N^2 Q_m^2}{r^4}\right) r^2 d\theta^2 \ + \ \left(1 - \ \frac{G_N^2 Q_m^2}{r^4}\right)^{-1} r^2 \sin^2\theta \ d\phi^2 \ . \end{aligned} \tag{18}$$

The generalized classical solution retains spherical symmetry when $Q_m = 0$. It possesses a curvature singularity at $r = 0$. On the other hand, the coordinate singularity turns out to become a scale dependent one and occurs either at $r_c = \sqrt{G_N Q_e}$ or at $r_s = 2G_N M$, which leads to a Schwarzschild black hole in the theory. The fact that the theory (10) allows two

semi-classical black holes (16) and (18) for $\mathcal{B} = 0$ may lead to an apparent puzzle. We shall see that the puzzle may be resolved by taking into account the $\mathcal{B}$-field in the formalism. The nonlinearity in the $U(1)$ gauge sector will be seen to modify its charges

$$Q^2_{\rm eff} = Q^2_e \left(1 - \frac{G_N|\mathbf{E}|}{Q_e} \ - \ \dots \right) \qquad \text{and} \qquad \bar{Q}^2_{\rm eff} = Q^2_m \left(1 - \frac{G_N|\mathbf{B}|}{Q_m} \ - \ \dots \right) \ . \tag{19}$$

Then, these two different semi-classical black hole geometries may be obtained from a generalized RN-black hole in a noncommutative string theory. For instance, the leading order term in eq.(19) would appropriately represent the RN-like black hole geometry (16) and the 1st order term there would govern the Schwarzschild like geometry (18). We postpone the detail of computations on the corrections to a later section 5.

2.3 Planckian regime

In the case, the space-time curvature on the D_3-brane becomes significant due to the nontrivial induced metric and gauge fields. The appropriate world-volume dynamics on a curved D_3-brane may be described at the Planck scale by an appropriate generalization of the action (9). The expansion under the square-root in the action (1) is performed by keeping track of a particular order in gauge field strength (say $\mathcal{F}^4$). Then, the complete action may be simplified to yield

$$\begin{aligned} S \ &= \ \int d^4y \ \sqrt{g} \ \left(\ \frac{1}{16\pi G_N} \ R \ - \frac{1}{4}\Big[\ \mathcal{F}^2 - \ \frac{1}{2}\mathcal{F}\mathcal{F}_+ \ F^2_- \ K^2(\mathcal{F}) \ \Big] \right) \\ &= \ \int d^4y \ \sqrt{g} \ \Bigg(\ \frac{1}{16\pi G_N} \ R \ - \frac{1}{4}\Big[\ \mathcal{F}^2 - \ \frac{1}{2}\mathcal{F}^4 \ K^2(\mathcal{F}) \\ &\qquad\qquad + \frac{1}{2}\Big(\mathcal{F}^3 \ {}^\star\mathcal{F} - \mathcal{F} \ {}^\star\mathcal{F}^3 + [\mathcal{F} \ {}^\star\mathcal{F}]^2\Big)K^2(\mathcal{F})\Big]\Bigg) \ , \end{aligned} \tag{20}$$

where $G_N = l_p^2$ signify the Planck scale in the theory. The field strengths $\mathcal{F}_\pm = (\mathcal{F} \pm {}^\star\mathcal{F})$ and $K(\mathcal{F})$ contains all higher order terms in field strength. The Hodge dual of $\mathcal{F}$ is denoted as ${}^\star\mathcal{F}$. Explicitly,

$${}^\star\mathcal{F}^{\mu\nu} = \frac{1}{2\sqrt{g}}\mathcal{E}^{\mu\nu\rho\lambda} \ \mathcal{F}_{\rho\lambda} \ , \tag{21}$$

where $\mathcal{E}^{\mu\nu\rho\lambda}$ is a covariant antisymmetric tensor density. However, the Minkowski's inequality in the theory (20) gives rise to the (anti-) self-duality condition

$$\mathcal{F}_{\mu\nu} = \pm{}^\star\mathcal{F}_{\mu\nu} \ . \tag{22}$$

It implies $E_i = B_i$ in the theory. In addition, the self duality can be seen to impose a lower bound on the DBI-action [7]. Since all the higher order terms in gauge fields vanish, the theory (20) yields an exact stringy description. Then the relevant action, on a curved D_3-brane, becomes

$$S = \int d^4y \ \sqrt{g} \ \Big(\ \frac{1}{16\pi G_N} \ R \ - \frac{1}{4}g^{\mu\lambda}g^{\nu\rho} \ \mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho} \ \Big) \ . \tag{23}$$

The Einstein's equations of motion (12), in the regime may be seen to be governed by the vacuum equations *i.e.* $T_{\mu\nu} = 0$. This is due to the self-duality condition on the nonlinear gauge field (22) in the theory. In other words, the Einstein's equation is not modified by the presence of a nonlinear Maxwell term on a curved D_3-brane. The remaining equation of motion is that for the gauge field and is given by

$$D_\mu \mathcal{F}^{\mu\nu} = 0 \ , \tag{24}$$

where $D_\mu \equiv (\partial_\mu - \frac{1}{2}\partial_\mu g^{\lambda\rho} g_{\lambda\rho})$. For convenience, we postpone the geometric detail on the D_3-brane to an appropriate section 5.

3 Effective noncommutative frame

3.1 Exact action in a 4-dimensional Euclidean space-time

In this section, we focus on the relevant noncommutative dynamics corresponding to a curved D_3-brane action (23). However, we begin with a flat D_3-brane as described in section 2.1. It is known that Seiberg-Witten map [19] transforms the ordinary gauge theory on the D_3-brane (1) to a noncommutative one. Though the induced fields g and $\mathcal{B}$ are constants, the invariant field strength $\mathcal{F}$ modifies the Einstein's metric non-trivially[4]. The flat brane picks up noncommutative geometry and is governed by an effective metric $G_{\mu\nu}$ (17). Then, the noncommutative D_3-brane dynamics may be obtained from the DBI-action (1) with appropriate modifications. With Riemannian signature ($\mu, \nu = 1, 2, 3, 4$), the relevant effective action becomes

$$\hat{S}_{\rm DBI} = T_D^{\rm nc} \int d^4y \ \left(\sqrt{G} - \ \sqrt{G + 2\pi\alpha' \hat{F}} \ \right) \ , \tag{25}$$

where $G \equiv \det G_{\mu\nu}$. The noncommutative $U(1)$ field strength on the world-volume is given by

$$\begin{aligned} \hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu \ , \ \hat{A}_\nu]_\star \\ &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu \ + \Theta^{\rho\lambda} \ \partial_\rho \hat{A}_\mu(x) \partial_\lambda \hat{A}_\nu(y)|_{x=y} + \ \mathcal{O}(\Theta^2) \ , \end{aligned} \tag{26}$$

where Θ is a noncommutative parameter in the theory. The effective theory on the noncommutative D_3-brane may be described similar to its counter-part (9) with ordinary geometry. The relevant action for a curved D_3-brane becomes

$$\begin{aligned} \hat{S} &= \frac{1}{16\pi G_N} \int d^4y \ \sqrt{G} \ \mathcal{R} \ + \ \hat{S}_{\rm DBI} \\ &= \int d^4y \ \sqrt{G} \left(\frac{1}{16\pi G_N} \mathcal{R} \ - \frac{1}{4} Tr\left(\hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right) \ - \mathcal{O}(\hat{F}^4) \ + \ \ldots \ \right) \ . \end{aligned} \tag{27}$$

The higher order terms starting with $\mathcal{O}(\hat{F}^4)$ essentially account for the stringy corrections in the theory. They may be worked out explicitly, using the Seiberg-Witten map, to yield

[4] Strictly, speaking the electromagnetic field modifies the string metric, which is related to the Einstein's metric by a conformal factor.

a relation between the noncommutative $U(1)$ gauge field and its ordinary counter-part. It is given by

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\Theta^{\nu\lambda} A_\nu \left(\partial_\lambda A_\mu + \mathcal{F}_{\lambda\mu}\right) + \ \mathcal{O}(\Theta^2)\ . \tag{28}$$

The corresponding field strengths are related as

$$\hat{F}_{\mu\nu} = \mathcal{F}_{\mu\nu} + \Theta^{\rho\lambda}\left(\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu\lambda} - A_\rho\partial_\lambda\mathcal{F}_{\mu\nu}\ \right)\ +\ \mathcal{O}(\Theta^2)\ . \tag{29}$$

Then, the gauge field corrections to the world-volume dynamics (27) may be computed. To $\mathcal{O}(\Theta)$, it may be expressed as

$$\begin{aligned}\hat{F}^4 + (\text{higher orders})\ \ \equiv\ \ -\frac{K^2(\mathcal{F})}{2}\Bigg[& \mathcal{F}_{\mu\nu}\mathcal{F}_+^{\mu\nu}\ \mathcal{F}_-^2 + \Theta^{\lambda\rho}\Big(\mathcal{F}_{\mu\lambda}\mathcal{F}_{\nu\rho}\mathcal{F}_+^{\mu\nu}\ \mathcal{F}_-^2 \\ & -A_\lambda\partial_\rho\mathcal{F}_{\mu\nu}\ \mathcal{F}_+^{\mu\nu}\ \mathcal{F}_-^2\ +\ (\mathcal{F}_+)_{\mu\lambda}(\mathcal{F}_+)_{\nu\rho}\ \mathcal{F}^{\mu\nu}\ \mathcal{F}_-^2 \\ & +2(\mathcal{F}_-)_{\mu\lambda}(\mathcal{F}_-)_{\nu\rho}\ (\mathcal{F}_-)^{\mu\nu}\ \mathcal{F}_{\delta\sigma}(\mathcal{F}_+)^{\delta\sigma} + \ldots\ \Big)\Bigg].\end{aligned} \tag{30}$$

The higher order terms show the presence of strong gauge curvature in the theory (27). Using self duality condition for the ordinary gauge field $\mathcal{F}_- = 0$, it is straight-forward to see that all the higher order terms vanish identically in a particular combination of the gauge field. Then the effective action (27) becomes exact to all orders in stringy corrections. It is given by

$$\hat{S} = \int d^4y\ \sqrt{G}\left(\frac{1}{16\pi G_N}\mathcal{R}\ -\frac{1}{4}G^{\mu\lambda}G^{\nu\rho}\ \hat{F}_{\mu\nu}\star\hat{F}_{\lambda\rho}\ \right)\ . \tag{31}$$

The Moyal $\star$-product in the noncommutative gauge theory is known to introduce nonlocal terms in the action due to the infinite number of derivatives there. The equations of motion, respectively, for the effective metric $G_{\mu\nu}$ and the noncommutative gauge potential $\hat{A}_\mu$ are given by

$$\begin{aligned} & \mathcal{R}_{\mu\nu} - \frac{1}{2}G_{\mu\nu}\mathcal{R} = (8\pi G_N)\,\hat{T}_{\mu\nu} \\ \text{and}\qquad & \Big[\delta_\mu^\lambda - \Theta^{\lambda\rho}\ \partial_\rho^y\hat{A}_\mu(y)\ +\ \mathcal{O}(\Theta^2)\ \ldots\Big]_{y\to x}\mathcal{D}_\lambda\hat{F}^{\mu\nu}(x) = 0\ ,\end{aligned} \tag{32}$$

where $\hat{T}_{\mu\nu}$ is the energy momentum tensor and $\mathcal{D}_\lambda = (\partial_\lambda - \frac{1}{2}\partial_\lambda G^{\rho\delta}G_{\rho\delta})$. We shall see that $\hat{T}_{\mu\nu} \neq 0$ in the effective theory (31), which is unlike to its ordinary counter-part (23).

On the other hand, the vanishing stringy contributions (30) may imply that the noncommutative theory (31) naturally corresponds to the gravity decoupling regime at Planck scale.[5] Then the **E**-field in its ordinary counter-part is equivalently described by $\mathbf{E} \to \mathbf{E}_c$. Since all the closed string modes decouple from the theory, the effective metric becomes $G_{\mu\nu} \to -(2\pi\alpha')^2[\mathcal{F}_{\mu\lambda}\delta^{\lambda\rho}\mathcal{F}_{\rho\nu}]$. Thus the effective dynamics, in the decoupling regime, is completely governed by the nonlinear electromagnetic field in the theory.

[5]It is known that the **E**-string drains off the stringy contribution and leads to a tension-less string in the effective theory.

3.2 Relevant modes of quantum gravity

In the effective theory of gravity on the D_3-brane, the noncommutative parameters (Θ_E and Θ_B) are fixed at any given energy scale in the theory. They take nonzero values in a wide range of energy including the classical regime. Since the radius of time-like brane coordinate is of the order of string length l_s [11], the noncommutativities $[y^0, y^{2,3}] = i\Theta_E^{2,3}$ imply large length scales, of the order of $l_\perp = |\Theta_E|/l_s$, along $y^{2,3}$. Similarly $[y^2, y^1] = -i\Theta_B^3$ and $[y^3, y^1] = i\Theta_B^2$ would enforce a string scale along y^1-coordinate in the theory. Two different length scales l_s and $l_\perp$ in the noncommutative frame-work [14], allow one to scale the 4-dimensional effective theory in terms of a parameter $\lambda = l_s/l_\perp << 1$. In other words, the noncommutative scaling is appropriately described by the limit $\lambda \rightarrow 0$. Since the induced fields $(g, \mathcal{B})$ are dimensionless in the frame-work, the scaling naturally introduces large and small dimensions in the theory. Under the scaling, the transverse ($\perp$-) coordinates $y^i \rightarrow y^i$ and the longitudinal (L-) ones $y^\alpha \rightarrow \lambda y^\alpha$ for ($\alpha = 1, 4$ and $i = 2, 3$). Thus within the geometrical setup, it is natural to consider a gauge choice $G_{i\alpha} = 0$ for the effective metric, *i.e.*

$$G_{\mu\nu} = \begin{pmatrix} \bar{h}_{\alpha\beta} & 0 \\ 0 & h_{ij} \end{pmatrix} . \tag{33}$$

$\bar{h}_{\alpha\beta}$ and h_{ij} represent the metric components, respectively, on the L- and $\perp$-spaces It is important to note that though the zero modes on the D_3-brane world-volume are noncommutative, its L- and $\perp$-spaces are independently described by the ordinary geometry, *i.e.* $\Theta^{\alpha\beta} = 0 = \Theta^{ij}$.

In the gauge (33), the effective action (27) may be re-expressed in terms of a noncommutative scaling parameter λ [25, 26, 27]. It is given by

$$\begin{aligned} \hat{S}_{\rm eff} \ &= \ \int d^2y^{(\alpha)}\ d^2y^{(i)} \sqrt{\bar{h}}\ \sqrt{h}\ \Big[\ \frac{1}{16\pi}\mathcal{R}_h + \frac{1}{64\pi}\ h^{ij}\ \partial_i\bar{h}_{\alpha\beta}\partial_j\bar{h}_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} \\ &\qquad + \frac{1}{16\pi\lambda^2}\Big(\mathcal{R}_{\bar{h}} + \frac{1}{4}\ \bar{h}^{\alpha\beta}\ \partial_\alpha h_{ij}\partial_\beta h_{kl}\epsilon^{ik}\epsilon^{jl}\Big) \\ &\qquad - \frac{1}{4}\Big(\frac{1}{\lambda^2}\ \bar{h}^{\alpha\beta}\bar{h}^{\gamma\delta}\ \hat{F}_{\alpha\gamma}\hat{F}_{\beta\delta}\ + \lambda^2\ \bar{h}^{ij}h^{kl}\ \hat{F}_{ik}\hat{F}_{jl} \\ &\qquad\qquad + 2\bar{h}^{\alpha\beta}h^{ij}\ \hat{F}_{\alpha i}\star\hat{F}_{\beta j}\Big)\ \Big] . \end{aligned} \tag{34}$$

In the scaling limit, $\lambda \rightarrow 0$, the on shell action becomes

$$\hat{S}_{\rm eff} = \int d^2y^{(\alpha)}d^2y^{(i)}\sqrt{\bar{h}}\ \sqrt{h}\ \Big[\frac{1}{16\pi}\mathcal{R}_h + \frac{1}{64\pi}\ h^{ij}\ \partial_i\bar{h}_{\alpha\beta}\partial_j\bar{h}_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} - \frac{1}{2}\bar{h}^{\alpha\beta}h^{ij}\ \hat{F}_{\alpha i}\star\hat{F}_{\beta j}\ \Big)\Big] . \tag{35}$$

The action is obtained by using the vacuum field configurations

$$\begin{aligned} &\partial_\alpha h_{ij} = 0\ , \\ &\mathcal{R}_{\bar{h}} = 0 \\ \text{and} \qquad &\hat{F}_{\alpha\beta} = 0\ . \end{aligned} \tag{36}$$

As pointed out, since the noncommutativity in the frame-work is utilized to obtain the small and large length scales, both L- and $\perp$-spaces in the frame-work are described by ordinary geometries. The general solutions to the equations of motion are given by

$$\begin{aligned} & h_{ij} \equiv h_{ij}(y_\perp) \ , \\ & \tilde{h}_{\alpha\beta} \equiv \partial_\alpha X^a \partial_\beta X^b \ \delta_{ab} \\ \text{and} \qquad & \hat{A}_\alpha = 0 \ , \end{aligned} \tag{37}$$

where $X^a(y_\perp)$ are arbitrary brane modes on a projected space for $a = (1, 2, 3, 4)$. In a static gauge $\bar{h}_{\alpha\beta} \rightarrow \delta_{\alpha\beta}$.

We learned that the 4-dimensional effective action (35) governs the gravitational dynamics in the $\perp$-space and that for the matter field in the L-space. The noncommutative matrix parameter $\Theta^{\mu\nu}$ in the frame-work act as a propagator in the internal space-time connecting the L- and $\perp$-spaces. The relevant equations of motion described by the $\perp$-space are given by

$$\begin{aligned} & (\mathcal{R}_h)_{ij} - \frac{1}{2} h_{ij} \mathcal{R}_h - \frac{1}{16} \Big(h_{ij} h^{kl} \partial_k \bar{h}_{\alpha\beta} \ \partial_l \bar{h}_{\gamma\delta} - 2 \partial_i h_{\alpha\beta} \ \partial_j \bar{h}_{\gamma\delta} \Big) \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} = (8\pi) \hat{T}_{ij} \\ \text{and} \quad & \Big[\delta^\lambda_\alpha - \Theta^{\lambda k} \partial^y_k \hat{A}_\alpha(y) \Big]_{y \rightarrow x} \mathcal{D}^\perp_\lambda \hat{F}^{\alpha i}(x) = 0 \ . \end{aligned} \tag{38}$$

Similarly for the L-space, the equations of motion are given by

$$\begin{aligned} & \frac{1}{2} \bar{h}_{\alpha\beta} \partial_i \bar{h}_{\lambda\rho} \ \partial^i \bar{h}_{\gamma\delta} \ \epsilon^{\lambda\gamma} \epsilon^{\rho\delta} + \Big(h_{kl} \ \partial_i h^{kl} \ \partial^i \bar{h}_{\gamma\delta} + 2 \partial_i \partial^i \bar{h}_{\gamma\delta} \Big) \epsilon^\gamma_\alpha \epsilon^\delta_\beta = (64\pi) \hat{T}_{\alpha\beta} \\ \text{and} \quad & \Big[\delta^k_i - \Theta^{k\lambda} \partial^y_\lambda \hat{A}_i(y) \Big]_{y \rightarrow x} \mathcal{D}^{\|}_k \hat{F}^{\alpha i}(x) = 0 \ . \end{aligned} \tag{39}$$

Using the general nature of the solutions (37), the equations of motion for $\hat{A}_i$ simplifies drastically to yield

$$\mathcal{D}^\perp_\alpha \partial^\alpha \hat{A}^i = 0 \ . \tag{40}$$

The energy momentum tensor (13) is computed in the noncommutative frame-work to yield $T_{\alpha i} = 0$. The remaining non-vanishing components turn out to yield

$$\begin{aligned} & \hat{T}_{\alpha\beta} = \frac{1}{2} \left[\frac{1}{2} \bar{h}_{\alpha\beta} \ h^{ij} \ \hat{F}_{\alpha' i} \star \hat{F}^{\alpha'}_j \ - \ h^{ij} \ \hat{F}_{\alpha i} \star \hat{F}_{\beta j} \ \right] \\ \text{and} \qquad & \hat{T}_{ij} = \frac{1}{2} \left[\frac{1}{2} h_{ij} \ h^{kl} \ \hat{F}_{\alpha k} \star \hat{F}^\alpha_l \ - \ \bar{h}^{\alpha\beta} \ \hat{F}_{\alpha i} \star \hat{F}_{\beta j} \ \right] \ . \end{aligned} \tag{41}$$

Since the $\perp$-components $\hat{T}_{ij}$ are weak in the frame-work, the energy momentum tensor in the theory may well be approximated by $\hat{T}_{\alpha\beta}$. It is straight-forward to check that $\hat{T}_{\alpha\beta}$ is traceless. Being a symmetric tensor, $\hat{T}_{\alpha\beta}$ is survived by two degrees of freedom in the theory. They may be identified with the left and right moving momenta of the noncommutative strings [14]. Using eq.(29), the relevant energy momentum tensor may be expressed in terms of the

ordinary gauge field strength. The self-duality condition (22) among the gauge fields in the theory simplifies the expression for $\hat{T}_{\alpha\beta}$. To $\mathcal{O}(\Theta)$, it becomes

$$\hat{T}_{\alpha\beta} = \frac{1}{2}\ \Theta^{k\lambda}\Big[\ \mathcal{F}_{\lambda i}\Big(\bar{h}_{\alpha\beta}\ \mathcal{F}^{i\rho}\mathcal{F}_{\rho k}\ +\ 2\mathcal{F}^{i}_{\alpha}\mathcal{F}_{k\beta}\ \Big) + A_k\Big(\bar{h}_{\alpha\beta}\ \mathcal{F}_{\rho i}\ \partial_\lambda\mathcal{F}^{i\rho}\ -\ 2\mathcal{F}_{\alpha i}\ \partial_\lambda\mathcal{F}^{i}_{\beta}\ \Big)\ \Big]\ . \qquad (42)$$

It confirms that the $\hat{T}_{\mu\nu}$ contributes significantly in the noncommutative frame-work. Since the L- and $\perp$-spaces are described independently by ordinary geometries in the effective theory, $T_{\mu\nu}$ and T_{ij} are traceless and hence $T_{\tilde{\tau}\tilde{\tau}} = -T_{\tilde{r}\tilde{r}}$.

4 Noncommutative scaling in $(\tilde{\tau}, \tilde{r}, \tilde{l}_\theta, \tilde{l}_\phi)$-coordinates

The scaling analysis discussed in the previous section is a consequence of space-time noncommutativity in the theory. Strictly speaking, it makes sense in the cartesian coordinate system. We use tilde notation in a noncommutative frame-work to distinguish them from their (ordinary) counter-part in the usual coordinates. For instance, we denote the noncommutative cartesian coordinates, $(\tilde{\tau}, \tilde{x}, \tilde{y}, \tilde{z})$ with Riemannian signature on the D_3-brane by $\tilde{\mathcal{J}}_C$-system. The limit $\Theta \to 0$, implies $\tilde{\mathcal{J}}_C \to \mathcal{J}_C$. On the other hand, it may be possible to approximate the "spherical polar" coordinates $(\tilde{\tau}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$ in a noncommutative set-up, *i.e.* $\tilde{\mathcal{J}}_S$-system. In particular, the noncommutative constraints derived from the $\tilde{\mathcal{J}}_C$-system may appropriately be incorporated in the $\tilde{\mathcal{J}}_S$-system.

Since the L- and $\perp$-spaces are described by two independent length scales, the 4-dimensional space-time may be approximated by two independent (2×2)-blocks, say $(\tilde{\tau}, \tilde{z})$ and $(\tilde{x}, \tilde{y})$. In particular, from a 4-dimensional point of view, the $(\tilde{\tau}, \tilde{z})$ space may be viewed with a fixed polar angle $\tilde{\theta}$ and then the remaining $(\tilde{x}, \tilde{y})$ space may be described with a fixed radial coordinate $\tilde{r}$. A priori, two sub-spaces in the theory are consistent with the total number space-time degrees of freedom. However, a careful analysis reveals that the noncommutative constraints further freeze some degrees of freedom in the theory.

4.1 S_2^Θ geometry

To begin with, consider the transformations, with ordinary geometries, between the cartesian (τ, x, y, z) and spherical polar coordinates (τ, r, θ, ϕ). They are given by

$$\tau \to \tau\ , \qquad z = r\cos\theta\ ,$$
$$x = r\sin\theta\cos\phi \ \ \text{and} \quad y = r\cos\theta\sin\phi\ . \qquad (43)$$

Let us incorporate the space-time noncommutative constraints in the $\mathcal{J}_C$-coordinate system to that in $\mathcal{J}_S$. Since, the angle θ may be kept fixed in the $(\tilde{\tau}, \tilde{z})$-coordinate sector, it leads to

$$d\tau^2 \to d\tilde{\tau}^2 \ \ \text{and} \qquad dz^2 \to d\tilde{r}^2 = dr^2\cos^2\theta\ , \qquad (44)$$

where $\tilde{r} = r\cos\theta > r$, for a nontrivial $0 < \theta < \pi$. Similarly $\tilde{r}$ is kept fixed in the $(\tilde{x}, \tilde{y})$ coordinate sector to yield a two sphere with a constant radius r_0. Then, the relevant line element in the L-sector becomes

$$(dx^2 + dy^2) \rightarrow (d\tilde{x}^2 + d\tilde{y}^2) = r_0^2 \left(\cos^2\theta\ d\theta^2 + \sin^2\theta\ d\phi^2\right)\ . \tag{45}$$

Since θ and ϕ are arbitrary angles in the $(\tilde{x}, \tilde{y})$-space sector, eq.(45) corresponds to a modified S_2 geometry. It implies that the spherical symmetry in a 4 dimensional space-time is broken in a noncommutative set-up, *i.e.* $S_2 \rightarrow S_2^{\Theta}$. The fact that S_2^{Θ} is defined with a constant radius $r = r_0$ is a consequence of independent L- and $\perp$-spaces, respectively, defined with small and large length scales in the theory. Since the above approximation becomes exact in the gravity decoupling limit [11], the effective description evolves with the notion of a two dimensional space-time in its semi-classical regime [14]. Then r_0 may be identified with the critical value of $\tilde{r}$, *i.e.* $r_0 = r_c$. Generalization of S_2^{Θ} line element (45) to that in a 4-dimensional effective theory possibly requires an arbitrary radial coordinate $\tilde{r} > r_c$. Then the relevant 4-dimensional Euclidean line element, obtained in the asymptotic limit, may be generalized from eqs.(44) and (45). In $\tilde{\mathcal{J}}_S$-coordinate system, the complete line element becomes

$$ds^2 = d\tilde{\tau}^2 + d\tilde{r}^2 + \tilde{r}^2\ d\Omega^2 - \tilde{r}^2 \sin^2\tilde{\theta}\ d\tilde{\theta}^2\ . \tag{46}$$

In a re-defined orthogonal noncommutative coordinate system $\tilde{\mathcal{J}}_o \equiv (\tilde{\tau}, \tilde{r}, \tilde{l}_\theta, \tilde{l}_\phi)$, the line element (46) becomes

$$ds^2 = d\tilde{\tau}^2 + d\tilde{r}^2 + \cos^2\tilde{\theta}\ d\tilde{l}_\theta^2 + d\tilde{l}_\phi^2\ , \tag{47}$$

$$\text{where} \qquad d\tilde{l}_\theta = \tilde{r}\ d\tilde{\theta} \qquad \text{and} \quad d\tilde{l}_\phi = \tilde{r}\ \sin\tilde{\theta}\ d\tilde{\phi}\ . \tag{48}$$

In addition to the correction in S_2 geometry, the radial coordinate is defined for large $\tilde{r}$, *i.e.* for $\tilde{r} > r_c$. Then the geometry in the limit $\Theta \rightarrow 0$ may be obtained from two limiting values on coordinates in $\tilde{\mathcal{J}}_O$, *i.e.* $\tilde{\theta} \rightarrow 0$ and $r_c \rightarrow 0$.

At this point, we recall the small radii obtained for the compact longitudinal coordinates following the noncommutative scaling on a D_3-brane [14]. Since the quantized longitudinal space in the frame-work is governed by a flat metric (37), it may well be described by the S_2^{Θ} geometry in $\tilde{\mathcal{J}}_S$. The quantum feature claimed to be associated with $(\tilde{\theta}, \tilde{\phi})$ coordinates is supported by the fact that the $\perp$-space in $\tilde{\mathcal{J}}_S$ is governed by large dimensions $(\tilde{r}, \tilde{\tau})$. Then the L-space governed by $(\tilde{l}_\theta, \tilde{l}_\phi)$ in $\tilde{\mathcal{J}}_O$ are, a priori, defined for infinitesimal $\delta\tilde{\theta}$ and $\delta\tilde{\phi}$, *i.e.* $(0 \leq \tilde{\theta} \leq \delta\tilde{\theta})$ and $(0 \leq \tilde{\phi} \leq \delta\tilde{\phi})$. In fact, the $\perp$-space may be seen to be parametrized by the plane polar coordinates $(\tilde{r}, \tilde{\tau})$ for $\tilde{r}\ \epsilon(r_c, \infty)$ and $\tilde{\tau}\ \epsilon[0, 4\pi\tau_c]$, where τ_c is a minimal noncommutative scale in the theory. A careful analysis in $\tilde{\mathcal{J}}_o$ leads to an equivalent coordinate system with a fixed angle $\tilde{\theta}$ and an arbitrary angle $\tilde{\phi}\ \epsilon[0, 2\pi]$. The resulting coordinate system

with a fixed $\tilde{\theta}$ is in precise agreement with our approximation to obtain the line element (47). In fact, the $\tilde{\mathcal{J}}_O$-system is consistent with the notion of light wedge instead of a light cone in a noncommutative theory [28].

4.2 Corrections to (τ, r)-coordinates

Now we look for an appropriate transformation from the noncommutative $\tilde{\mathcal{J}}_S$ system to its ordinary counter-part in the spherical polar coordinates (τ, r, θ, ϕ), *i.e.* $\mathcal{J}_S$-system. At this point, we recall the transformations obtained between the zero modes on a D_3-brane in ref.[14]. It has been argued that the transformations incorporate (small) constant shifts on the brane coordinates and make them discrete. Then the noncommutative D_3-brane (zero modes) coordinates are given by

$$\tilde{y}^{\mu} = y^{\mu} + \Theta^{\mu\nu} p_{\nu} \ , \tag{49}$$

where p_{μ} is the canonical conjugate momentum to y^{μ}. Explicitly, the shift transformations are worked out, in $\tilde{\mathcal{J}}_C$-system, to yield

$$\begin{aligned} \tilde{\tau} \ &= \ \tau + (\mathbf{\Theta}_E \cdot \mathbf{p}) \\ \text{and} \quad \tilde{r}^i \ &= \ r^i + \epsilon^{ijk} \Theta_B^k \ p_j + \Theta_E^i \ p_4 \ , \end{aligned} \tag{50}$$

where $\Theta^{4i} \equiv \Theta_E^i = (0, \Theta_E^y, \Theta_E^z)$ and $\Theta_B^i = (0, \Theta_B^y, \Theta_B^z)$. Then $\tilde{r}^2$ becomes

$$\tilde{r}^2 = r^2 + r_c^2 \ . \tag{51}$$

Explicitly, r_c^2 is given by

$$\begin{aligned} r_c^2 \ &= \ (\mathbf{\Theta}_B \cdot \mathbf{L}) \ + \ (\mathbf{\Theta}_E \cdot \mathbf{r}) p_4 \ + \ \mathcal{O}(\Theta^2) \ + \ \dots \ , \\ &\equiv \ \Theta + \mathcal{O}(\Theta^2) \ + \ \dots \ , \end{aligned} \tag{52}$$

where $\mathbf{L}$ denotes the angular momentum vector and $|\mathbf{\Theta_B}| = |\mathbf{\Theta_E}|$. It shows that the every coordinate in the corresponding cartesian system $\tilde{\mathcal{J}}_C$ are bounded from below. In fact the bound on the radial coordinate, *i.e.* r_c, arises from the upper bound on the $\mathbf{E}$ field in the theory.

On the other hand, the transformation between $\tilde{\mathcal{J}}_C \to \tilde{\mathcal{J}}_O$ may equivalently be viewed as an orthogonal transformation between its commutative counter-parts. In particular, it is given by an orthogonal rotation $\mathcal{R}_\theta$ of (r, θ)-coordinates by an arbitrary polar angle θ around the symmetry axes $(\tau, \, \phi)$. The transformations may be given by

$$\mathcal{J}_C \to \mathcal{J}_S \equiv \tilde{\mathcal{J}}_O + \mathcal{R}_\theta \ . \tag{53}$$

It implies that there are several coordinate systems equivalent to $\tilde{\mathcal{J}}_O$ and they are defined for different values θ in the range $\theta \ \epsilon (0, \pi)$. In other words, the continuous coordinate θ in the

S_2 geometry, picks up discrete values in its noncommutative counter-part S_2^{Θ}. As a result the spherical symmetry around an axis is broken in the theory. However the configuration space involving all possible rotations $\mathcal{R}_\theta$, preserves the spherical symmetry in the noncommutative frame-work. Thus the inclusion of configuration space for θ, in $\tilde{\mathcal{J}}_O$, results in a spherically symmetric coordinate system equivalent to $\mathcal{J}_S$.

5 Generalized black holes

5.1 Schwarzschild like geometries

In this section, we construct a generalized black hole geometry in the effective noncommutative theory (35). In particular, we attempt to formulate the Schwarzschild like geometry (18) obtained, with an ordinary space-time, to the noncommutative one. Since the effective metric (17) is used to obtain the Schwarzschild like black hole, the underlying theory is naturally governed by the noncommutative formalism at Planck energy.

To begin with, we rather focus on the ordinary counter-part (23) of the noncommutative string theory (31). The self-duality in the gauge field yields $Q_e = Q_m = Q$. However the energy momentum tensor $T_{\mu\nu} = 0$ in the theory. Thus the relevant dynamics is essentially described by the vacuum Einstein's equations of motion. Its solution is given by the Schwarzschild black hole. In addition, the gauge potential (14) in the theory consistently describes a parallel field configuration, *i.e.* $\mathbf{E} = (E_{\hat{r}}, 0, 0)$ and $\mathbf{B} = (B_{\hat{r}}, 0, 0)$. The EM-field is obtained (15) may be be checked to be in agreement with the duality invariance (21) in the theory.

Now a generalized semi-classical solution on a curved D_3-brane (27), may be constructed using the effective metric (17) in the theory. Interestingly for $\mathcal{B} = 0$, the semi-classical solution has been obtained in eq.(18). However, in presence of $\mathcal{B}$-field, the effective metric in $\tilde{\mathcal{J}}_O$-system may be expressed as

$$\begin{aligned} ds^2 \;=\; & \left(1 - \frac{2G_N M}{\tilde{r}} - \frac{(G_N Q)^2}{\tilde{r}^4} + \frac{2G_N^3 M Q^2}{\tilde{r}^5}\right) d\tilde{\tau}^2 \\ & + \left(1 - \frac{2G_N M}{\tilde{r}} - \frac{(G_N Q)^2}{\tilde{r}^4} + \frac{2G_N^3 M Q^2}{\tilde{r}^5}\right)^{-1} d\tilde{r}^2 \\ & + \left(1 - \frac{(G_N Q)^2}{\tilde{r}^4}\right) d\tilde{l}_\theta^2 + \left(1 - \frac{(G_N Q)^2}{\tilde{r}^4}\right)^{-1} d\tilde{l}_\phi^2 \ , \end{aligned} \tag{54}$$

The generalized solution breaks the spherical symmetry, which is due to the fact that $\tilde{\theta}$ is fixed in $\tilde{\mathcal{J}}_O$-coordinate system. The gauge potential $\hat{A}_\mu$ is worked out, from eq.(28) using eq.(14), in the theory. Since the constraints (37) ensure $\hat{A}_{\tilde{\phi}} = 0$, the generalized solution (54) may be seen to be governed by a trivial noncommutative gauge potential $\hat{A}_\mu = 0$.

The curvature singularity in the generalized solution (54) appears at $\tilde{r} = r_c$. The solution possesses a coordinate singularity at $r_c = \pm\sqrt{G_N Q}$ in addition to its event horizon at $r_s = 2G_N M$, where $r_s > r_c$. Since $\tilde{r}$ possesses a lower bound at $\tilde{r} = r_c$, the curvature singularity in the solution (54) is governed only at its lower bound. In other words, the solution (54) describes a GS-black hole geometry in the effective theory. Though the GS-solution possesses both (equal) electric and magnetic charges, its event horizon is actually governed by its Schwarzschild radius r_s. This is due to the underlying fact that $T_{\mu\nu} = 0$ in the theory. On the other hand, the exact nature (in Θ) of the effective theory makes it accessible to its electric charge Q at high energy. Thus Q adds a small mass and may be seen to be associated with the angular momentum (52) in the effective noncommutative theory.

The GS-black hole (54) on a noncommutative D_3-brane, essentially describes an Euclidean manifold $R^2 \times S^2$ and is valid for large $(\tilde{r}, \tilde{\tau})$, *i.e.* for $\tilde{r} \geq r_c$ and $\tilde{\tau} \geq \tau_c$. Since the $\perp$-space is spanned by the large dimensions in a noncommutative frame-work [11, 14], they may be well described by $(\tilde{r}, \tilde{\tau})$ in the frame-work. The remaining two orthogonal coordinates $(l_{\tilde{\theta}}, l_{\tilde{\phi}})$ there describe the quantized L-plane.

Using eq.(51), the relations between the parameters in the two coordinate systems $\tilde{\mathcal{J}}_O$ and $\mathcal{J}_S$ are worked out to yield

$$\begin{aligned} M_{\text{eff}} &= (G_N M)\left[1 - \frac{\Theta}{2r^2} + \mathcal{O}(\Theta^2) + \ldots\right] \\ \text{and} \qquad r_c^2 &= (G_N Q)\left[1 - \frac{\Theta}{r^2} + \mathcal{O}(\Theta^2) + \ldots\right] . \end{aligned} \tag{55}$$

Finally, the generalized geometry (54), in $\mathcal{J}_S$-coordinates, becomes

$$\begin{aligned} ds^2 \ = \ & \left(1 - \frac{2M_{\text{eff}}}{r} - \frac{r_c^4}{r^4} + \frac{2M_{\text{eff}}\ r_c^4}{r^5}\right) d\tau^2 + \left(1 - \frac{2M_{\text{eff}}}{r} - \frac{r_c^4}{r^4} + \frac{2M_{\text{eff}}\ r_c^4}{r^5}\right)^{-1} dr^2 \\ & + \left(1 - \frac{r_c^4}{r^4}\right) r^2 d\theta^2 \ + \ \left(1 - \frac{r_c^4}{r^4}\right)^{-1} r^2 \sin^2\theta d\phi^2 \ . \end{aligned} \tag{56}$$

The curvature singularity in the GS-solution occurs at $r = 0$ in $\mathcal{J}_S$-system. In addition, the geometry possesses two distinct coordinate singularities at $r = r_c$ and r_s. Since $r_c < r_s$, the curvature singularity of the semi-classical black hole is well protected by its event horizon at r_c in the frame-work. The other coordinate singularity at r_s is in the classical regime. It may not serve the purpose of an event horizon and hence we call it a "pseudo horizon" in the theory. In fact, the GS-geometry at its Schwarzschild radius possesses all properties of that of an event horizon for $M \neq 0$. The possible cause of hindrance at r_s to treat it as an event horizon may be resolved by the fact that the D_3-brane is flat in the classical regime, *i.e.* $M = 0$ on its world-volume. Then an observer's motion on a D_3-brane world-volume appears to be "tangential" to

the event horizon of the classical GS-black hole. However the inherent noncommutative scale in the theory deform the tangential touch of the horizon to a finite small cut on the D_3-brane. Heuristically, the observer is rather just inside or just outside the event horizon r_s of a typical Schwarzschild black hole. At this point, we recall an e^-e^+ pair production process, just outside r_s, initiated by the uniform **E**-field on the D_3-brane in the classical regime. If an e^- moves in the out-ward direction, then the e^+ would traverse the coordinate singularity at r_s in the in-ward direction. Subsequently the e^+ behaves like an e^- inside r_s and its energy becomes negative, which is required by the energy conservation in the pair production process. Since the event horizon on the D_3-brane is actually governed at r_c, the observer would experience an effective gravitational force due to the **E**-field in the classical regime.

Further more, the pair creation phenomenon by an uniform **E**-field may be seen to explain the Hawking radiation [2] from the generalized GS-geometry (56). In the context, we find that eq.(55) shows the presence of effective parameters, $M_{\rm eff}$ and r_c in the generalized black hole in a noncommutative formalism. Since Θ-terms modify the mass of the black hole, its Schwarzschild radius $r_s = 2M_{\rm eff}$ is not fixed in the $\mathcal{J}_S$-system. It confirms that the semi-classical GS-black hole Hawking radiates. The source of radiation is primarily due to the Θ-terms, which give rise to the nonlinearity in the **E**-field. Since the nonlinearity is determined by the non-vanishing global mode of the **E**-field, the source of radiation is uniform in the frame-work. An e^-e^+ pair created by an uniform **E**-field at the vacuum solution to the Einstein's equations of motion may be viewed as that produced just outside r_s. As pointed out, an e^- motion in the out-ward direction from r_s would associate the e^+ motion in the in-ward direction. Since the velocity of a particle is zero at r_s, the motion of the negative energy e^+ may be viewed as that of a positive energy (secondary) e^- in the out-ward direction. However, the secondary e^- is time dilated as it overcomes the horizon from inside. The pair production continues with the increase of **E**-field, at each step, until $\mathbf{E}_C$ is reached. The e^+ from the second pair produced at r_s is attracted by the dilated e^-, they annihilate to produce the quantum of radiation in the theory. In each step, the event horizon at r_s shrinks in its size. The Hawking temperature for the GS-black hole is worked out to yield

$$T^{\rm GS}_{\rm Hawking} = \frac{1}{8\pi M_{\rm eff}}\ . \tag{57}$$

It implies that the temperature increases in each step, followed by the quantum radiation, in the theory.

On the other hand, the higher order corrections, in $1/r$ to the Einstein's metric dominate at the Planck scale, *i.e.* in the limit $\mathbf{E} \rightarrow \mathbf{E}_c$. They are indeed associated with the electric charge in the theory. The correction term, *i.e.* Q^2/r^4, to the Einstein's metric $g_{\mu\nu}$ may be identified with the energy momentum tensor computed in a semi-classical theory governed by the Einstein's

theory coupled to the Maxwell's. In other words, they contribute a mass correction (55) to the Schwarzschild mass M otherwise present in the theory. At Planck scale, since Q^2/r^4 dominates over $2M_{\rm eff}/r$, the 4-dimensional generalized geometry (56) may alternately be viewed as a higher dimensional, semi-classical, Schwarzschild black hole produced in the scattering of noncommutative strings [14]. In particular an d-dimensional Schwarzschild black hole mass is known to be associated with $r^{(3-d)}$ term in the metric. Then the semi-classical black hole produced in a high energy collision may naively be argued to have its origin in 7-dimensions. However the noncommutative constraints in the theory reduce the space-time dimensions to five from seven. Interestingly, the effective 4-dimensional space-time (56) may be obtained from a 5-dimensional gravity (*eg.* see [29, 30]) with a small scale along the fifth dimension. The generalized black hole (56) in an effective theory of gravity is identical to that obtained in the Einstein's GTR. However in the classical regime, the corrections in the effective metric are small and may be ignored. Then, the generalized geometry (56) precisely reduces to the Schwarzschild black hole.

5.2 Reissner-Nordstrom like geometries

Let us recall the approximations enforced on the $\tilde{\mathcal{J}}_S$-coordinates, which in turn lead to a more appropriate $\tilde{\mathcal{J}}_O$-system on the noncommutative D_3-brane. Since $\tilde{\theta} \rightarrow 0$ in $\tilde{\mathcal{J}}_O$-system, an equivalent arbitrary θ description (53) in $\mathcal{J}_S$-coordinates give rise to useful insights into the theory.

Now let us consider the the general solution (37) in $\tilde{\mathcal{J}}_O$-system. The potential in its general form becomes

$$\hat{A}_\mu = \left(\hat{A}_{\tilde{\tau}}, \hat{A}_{\tilde{r}}, 0, 0\right) \ . \tag{58}$$

Eq.(40) confirms that the components of gauge field in $\tilde{\mathcal{J}}_O$ coordinates satisfy

$$\begin{aligned} &\left(\partial^2_{\tilde{l}_\theta} + \partial^2_{\tilde{l}_\phi}\right)\hat{A}_{\tilde{\tau}} = 0 \\ \text{and} \qquad &\left(\partial^2_{\tilde{l}_\theta} + \partial^2_{\tilde{l}_\phi}\right)\hat{A}_{\tilde{r}} = 0 \ . \end{aligned} \tag{59}$$

Then the noncommutative gauge potential becomes trivial in $\tilde{\mathcal{J}}_O$ coordinates. However in $\tilde{\mathcal{J}}_S$-coordinates, *i.e.* for arbitrary $\tilde{\theta}$, the non-vanishing orthogonal components of the potential are given by

$$\hat{A}_{\tilde{\tau}} = -i\hat{Q}\ \sin\tilde{\theta}\cos\tilde{\phi} \quad \text{and} \qquad \hat{A}_{\tilde{r}} = \hat{Q}\ \sin\tilde{\theta}\sin\tilde{\phi}\ , \tag{60}$$

where $\hat{Q}$ is a constant and may be interpreted as an electric charge in the noncommutative theory. Then the orthogonal components of the EM-field, in $\tilde{\mathcal{J}}_S$-system, are computed to yield

$$E_{\hat{\theta}} = \frac{\partial_\theta \hat{A}_0}{\tilde{r}\cos\tilde{\theta}} = \frac{\hat{Q}}{\tilde{r}}\cos\tilde{\phi}\ ,$$

$$
E_{\hat{\phi}} = \frac{\partial_\phi \hat{A}_0}{\tilde{r}\sin\tilde{\theta}} = -\frac{\hat{Q}}{\tilde{r}}\sin\tilde{\phi}
$$

and

$$
\begin{aligned}
B_{\hat{\theta}} &= \frac{\partial_{\hat{\phi}} \hat{A}_{\hat{r}}}{\tilde{r}\sin\tilde{\theta}} = \frac{\hat{Q}}{\tilde{r}}\cos\tilde{\phi}\ , \\
B_{\hat{\phi}} &= -\frac{\partial_{\hat{\theta}} \hat{A}_{\hat{r}}}{\tilde{r}\cos\tilde{\theta}} = -\frac{\hat{Q}}{\tilde{r}}\sin\tilde{\phi}\ .
\end{aligned} \tag{61}
$$

The **E**- and **B**-field components are satisfy the duality invariance (21) in the theory. In fact, they (61) describe a parallel EM-field in $\tilde{\mathcal{J}}_O$-coordinates. The energy momentum tensor (42) in the theory is computed to yield

$$
\begin{aligned}
\hat{T}_{\tau\tau} &= -\ \frac{i\hat{Q}^3\Theta_E}{\tilde{r}^3}\left[\cos\tilde{\phi} - \sin\tilde{\phi}\right] \\
&= -\frac{i\hat{Q}^2}{\tilde{r}^2}\left[\cos\tilde{\phi} - \sin\tilde{\phi}\right]\ .
\end{aligned} \tag{62}
$$

It implies that the limit $\Theta_E \to 0$, for an arbitrary angle $\tilde{\phi}$, may alternately be described by $\phi = \pi/4$ for a fixed Θ. On the other hand, the **E**- and **B**-fields (61) may be incorporated into the modified metric (17) for a vacuum solution to Einstein's theory. In the $\tilde{\mathcal{J}}_O$-coordinates, the effective solution becomes

$$
\begin{aligned}
ds^2 \ =\ & \left(1 - \frac{2G_NM}{\tilde{r}} - \frac{G_N\hat{Q}^2}{\tilde{r}^2}\right)d\tilde{\tau}^2 + \left(1 - \frac{2G_NM}{\tilde{r}} - \frac{G_N\hat{Q}^2}{\tilde{r}^2}\right)^{-1} d\tilde{r}^2 \\
& + \left(1 - \frac{G_N\hat{Q}^2}{\tilde{r}^2}\cos 2\tilde{\phi} - \frac{2G_N^2M\hat{Q}^2}{\tilde{r}^3}\right)d\tilde{l}_\theta^2 \\
& + \left(1 + \frac{G_N\hat{Q}^2}{\tilde{r}^2}\cos 2\tilde{\phi} - \frac{2G_N^2M\hat{Q}^2}{\tilde{r}^3}\right)d\tilde{l}_\phi^2\ .
\end{aligned} \tag{63}
$$

The generalized geometry possesses a curvature singularity at $\tilde{r} = \hat{Q}$. The solution describes a GRN-black hole in a noncommutative open string theory with event horizons at

$$
r_\pm = (G_NM) \pm \sqrt{(G_NM)^2 - G_N\hat{Q}^2}\ . \tag{64}
$$

Thus the noncommutative D_3-brane world-volume geometry may also be governed by a GRN-black hole (63), in addition to the GS-geometry (56). Using eq.(51), we obtain

$$
\begin{aligned}
\hat{Q}^2_{\text{eff}} &= (G_N\hat{Q}^2)\left[1 - \frac{\Theta}{r^2} + \mathcal{O}(\Theta^2) + \ldots\right] \\
&= (G_N\hat{Q}^2) - (G_N\hat{Q}^2_\Theta)\left[1 - \frac{\Theta}{r^2} + \mathcal{O}(\Theta^2) + \ldots\right]\ ,
\end{aligned} \tag{65}
$$

where $\hat{Q}_\Theta$ is a non-zero constant in the theory. Then the GRN-black hole (63), in (τ, r, θ, ϕ) coordinates may be re-expressed as

$$
ds^2 \ =\ \left(1 - \frac{2M_{\text{eff}}}{r} \pm \frac{\hat{Q}^2_{\text{eff}}}{r^2}\right)d\tau^2 + \left(1 - \frac{2M_{\text{eff}}}{r} \pm \frac{\hat{Q}^2_{\text{eff}}}{r^2}\right)^{-1} dr^2
$$

$$
\begin{aligned}
&+\left(1\pm\frac{\hat{Q}_{\rm eff}^2}{r^2}\cos 2\phi\pm\frac{2M_{\rm eff}\hat{Q}_{\rm eff}^2}{r^3}\right)r^2d\theta^2\\
&+\left(1\mp\frac{\hat{Q}_{\rm eff}^2}{r^2}\cos 2\phi\pm\frac{2M_{\rm eff}\hat{Q}_{\rm eff}^2}{r^3}\right)r^2\sin^2\theta\ d\phi^2\ . \qquad (66)
\end{aligned}
$$

As expected in a noncommutative theory, the black hole geometry is not spherically symmetric. It reconfirms that for $\phi=\pi/4$, the S_2 symmetry is restored which is equivalently described by the $\Theta\rightarrow 0$.

Semi-classical GRN-geometry

The GRN-solution (66) in the regime is described by the (+)ve sign in its metric components. The semi-classical GRN-solution may be seen to possess a curvature singularity at $r=0$. It describes a charged black hole with event horizons at

$$
r_{\pm}=M_{\rm eff}\pm\sqrt{M_{\rm eff}^2-\hat{Q}_{\rm eff}^2}\ . \qquad (67)
$$

Since the mass $M_{\rm eff}$ and charge $\hat{Q}_{\rm eff}$ are not constants, the semi-classical GRN-black hole exhibits Hawking radiation. The Hawking temperature may be computed to yield

$$
T_{\rm Hawking}^{\rm GRN}=\frac{\sqrt{M_{\rm eff}^2-\hat{Q}_{\rm eff}^2}}{2\pi\left[M_{\rm eff}+\sqrt{M_{\rm eff}^2-\hat{Q}_{\rm eff}^2}\right]^2}\ . \qquad (68)
$$

In the limit $\Theta\rightarrow 0$, the GRN-geometry in the regime corresponds to the Reissner-Nordstrom (RN-) like geometry. It is given by

$$
\begin{aligned}
ds^2\ &=\ \left(1-\frac{2M}{r}+\frac{G_N\hat{Q}^2}{r^2}\right)d\tau^2+\left(1+\frac{2M}{r}+\frac{G_N\hat{Q}^2}{r^2}\right)^{-1}dr^2\\
&+\left(1+\frac{G_N\hat{Q}^2}{r^2}\cos 2\phi\right)r^2d\theta^2\ +\ \left(1-\frac{G_N\hat{Q}^2}{r^2}\cos 2\phi\right)r^2\sin^2\theta\ d\phi^2\ . \qquad (69)
\end{aligned}
$$

In the limit $\phi=\pi/4$, then the solution precisely describes the RN-black hole goemetry. It may be seen to exhibit Hawking radiation in the frame-work. The Hawking temperature is given by

$$
T_{\rm Hawking}^{\rm RN}=\frac{\sqrt{M^2-G_N^1\hat{Q}^2}}{2\pi\left[M+\sqrt{M^2-G_N^{-1}\hat{Q}^2}\right]^2}\ . \qquad (70)
$$

The temperature decreases with the quantum radiations from the semi-classical RN-black hole. With the emission of radiation, M decreases and finally becomes equal to its charge $\hat{Q}/\sqrt{G_N}$. The process ceases as $T_{\rm Hawking}^{\rm GRN}\rightarrow 0$ in the limit $M\rightarrow\hat{Q}/\sqrt{G_N}$, which may be identified with the gravity decoupling limit in the theory. In other words, in the limit of vanishing nonlinearity

$\mathcal{B} = 0$, the GRN-geometry in the classical regime reduces to a typical RN-black hole, which is described by the linear **E**-field. Schematically, the noncommutative D_3-brane geometries in $\tilde{\mathcal{J}}_O$-system for various values of r is illustrated in fig.2.

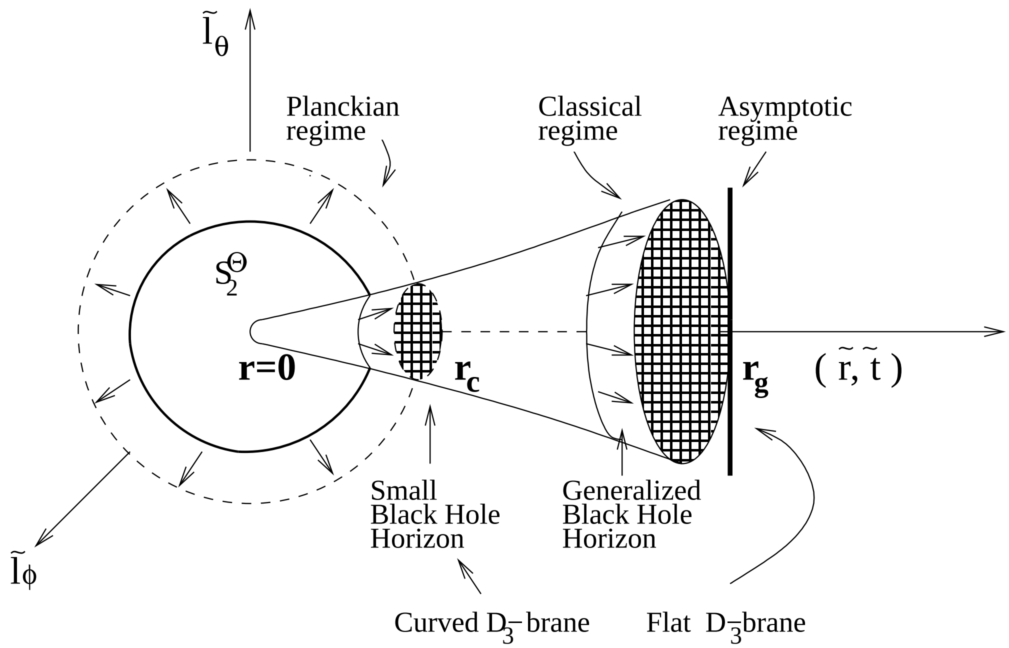

Figure 2: Noncommutative D_3-brane (sphere) geometries primarily at two different length scales l_s and $l_\perp$. They are characterized by Planckain ($r \to r_c$), classical ($r_c < r \leq r_g$) and asymptotic ($r_g \leq r \leq \infty$) regimes. The curved D_3-brane is defined for $r \leq r_g$ and the flat brane is defined in the asymptotic regime there.

GRN-geometry at Planck scale

The $(-)$ve sign in the GRN-geometry (66) leads to an apparent contradiction to the nature of a Reissner-Nordstrom (RN-) like black hole. The Hawking temperature in the regime becomes

$$T^{\rm GRN}_{\rm Hawking} = \frac{\sqrt{M^2_{\rm eff} + \hat{Q}^2_{\rm eff}}}{2\pi \left[M_{\rm eff} + \sqrt{M^2_{\rm eff} + \hat{Q}^2_{\rm eff}}\right]^2}\ . \tag{71}$$

The temperature increases with the emission of quantum radiations from the GRN-black hole, which is unlike to that of a typical semi-classical RN-black hole. However, the contradiction may be resolved by the fact that the noncommutative theory is scale dependent. In the regime, the GRN-solution is governed by a GS-geometry, which is evident from the increase in Hawking temperature (71) in the theory. The phenomenon is supported by the fact that the GRN-black hole becomes an extremum in the regime. GS-geometry becomes obvious when two black holes

(56) and (66) are compared with respect to their metric components. Since the higher order terms in $1/r$ become significant in the regime, the GRN-geometry is effectively governed by the GS-black hole. For instance a leading order term, in Θ, present in the GS-geometry may be seen to arise from the linear order term in the GRN-black hole and so on.

6 Two dimensional black holes

6.1 Semi-classical geometry

Now let us focus on the near horizon geometry in the semi-classical regime obtained from the GRN-black hole (66). Explicitly it is given by

$$\begin{aligned} ds^2 \ &= \ \left(1-\frac{2M_{\rm eff}}{r}+\frac{\hat{Q}_{\rm eff}^2}{r^2}\right)d\tau^2+\left(1-\frac{2M_{\rm eff}}{r}+\frac{\hat{Q}_{\rm eff}^2}{r^2}\right)^{-1}dr^2 \\ &\qquad + \left(1+\frac{\hat{Q}_{\rm eff}^2}{r^2}\cos 2\phi+\frac{2M_{\rm eff}\hat{Q}_{\rm eff}^2}{r^3}\right)r^2d\theta^2 \\ &\qquad + \left(1-\frac{\hat{Q}_{\rm eff}^2}{r^2}\cos 2\phi+\frac{2M_{\rm eff}\hat{Q}_{\rm eff}^2}{r^3}\right)r^2\sin^2\theta\ d\phi^2\ . \end{aligned} \tag{72}$$

The $\perp$-space may be seen to be redefined by a set of polar coordinates (ρ,ω) for $0\leq\rho\leq\infty$ and $0\leq\omega\leq 2\pi$. We define

$$\left(1-\frac{2M_{\rm eff}}{r}\right)=\frac{\rho^2}{8M_{\rm eff}}\ , \tag{73}$$

then in the limit $r\rightarrow r_s(=2M_{\rm eff})$, the radial coordinate becomes $\rho\rightarrow 0$. Since $\hat{Q}_{\rm eff}^2/r^2\rightarrow 0$ in the limit, the geometry reduces to that of a GS-black hole. Then, the semi-classical GRN-black hole (66) is simplified for its near horizon geometry to yield an Euclidean Rindler space-time in the $\perp$-plane. On the other hand, the L-space there, describes the high energy modes and decouples completely in the limit $r\rightarrow r_s$. The near horizon geometry in GRN-black hole in the semi-classical regime may be approximated by

$$ds^2=d\rho^2+\frac{4\rho^2}{r_s^2}d\tau^2+8r_s^2\ d\Omega^2\ . \tag{74}$$

It shows that the $\perp$- and L-spaces are completely independent in the near horizon geometry of the semi-classical GRN-black hole. In the regime, the effective space-time is governed by the $\perp$-space is given by

$$ds_{\perp}^2=d\rho^2+\rho^2d\omega^2\ , \tag{75}$$

where $\omega=2\tau/r_s$. Then, the Euclidean time becomes $\tau\rightarrow\tau+2\pi M_{\rm eff}$. It implies that the radius of the Euclidean time like coordinate is large in the near horizon geometry. On the other hand, the L-space in the effective 4-dimensional frame-work is described by the S_2 geometry. Thus the relevant near horizon semi-classical GRN geometry on a noncommutative D_3-brane is governed by a two dimensional black hole (75) along its $\perp$-space.

With an appropriate re-parametrization, *i.e.* $\rho = \sinh r$, the semi-classical black hole solution (75) may be seen to describe Witten's $2D$ black hole [31]

$$ds_{\perp}^2 = dr^2 - \tanh^2 r \; dt^2 \; . \tag{76}$$

The black hole geometry is known to be governed by an exact conformal field theory. The event horizon of the $2D$-black hole (76) is at $r = 0$. The scalar curvature, $R_h = 4/\cosh^2 r$, remains regular at the event horizon.

6.2 Gravity decoupled regime

The gravity decoupling limit in the effective noncommutative theory (35) is described by the decoupling of all the closed string modes in the theory. The limit may be incorporated into the theory by simply taking $M \rightarrow 0$, which is equivalently represented by $M \rightarrow \hat{Q}/\sqrt{G_N}$. In other words, the limit leads to the Planckian regime in the frame-work. It is interesting to see that the GRN-geometry (66) in the limit reduces to an extremum GRN-black hole, which in turn may be identified with the GS-black hole solution (56). In other words, the D_3-brane geometry in the Planckian regime is governed uniquely by the Schwarzschild like black hole.

To begin with, we consider the GS-black hole (56) in the gravity decoupling limit, *i.e* $r \rightarrow r_c$. Then the reduced geometry may be approximated by

$$ds^2 = \left(1 - \frac{r_c^4}{r^4}\right)\left[d\tau^2 + r^2 d\theta^2\right] + \left(1 + \frac{r_c^4}{r^4}\right)\left[dr^2 + r^2 \sin^2\theta d\phi^2\right] \; . \tag{77}$$

It describes a 4-dimensional semi-classical Schwarzschild black hole in a gravity decoupled noncommutative string theory. The black hole (77) possesses a small mass r_c and its event horizon is at $r = r_c$. It describes a laboratory black hole produced in an intermediate state of the Planckian energy scattering phenomenon investigated in ref.[14].[6]

On the other hand, the GRN-black hole (66), in the gravity decoupling limit $r \rightarrow \hat{Q}_{\text{eff}}$ may be worked out to yield

$$\begin{aligned} ds^2 = \left(1 - \frac{\hat{Q}_{\text{eff}}^2}{r^2}\right) d\tau^2 + \left(1 + \frac{\hat{Q}_{\text{eff}}^2}{r^2}\right) dr^2 + \left(1 - \frac{\hat{Q}_{\text{eff}}^2}{r^2}\cos 2\phi\right) r^2 d\theta^2 \\ + \left(1 + \frac{\hat{Q}_{\text{eff}}^2}{r^2}\cos 2\phi\right) r^2 \sin^2\theta \; d\phi^2 \; . \end{aligned} \tag{78}$$

The geometry describes a 4-dimensional Euclidean black hole with event horizon at $r = \pm\hat{Q}_{\text{eff}}$. The black hole possesses a small mass due to the nonlinear electromagnetic field and may be seen to describe the laboratory black hole (77).

[6]Interestingly, these laboratory black holes production have drawn considerable attention in the low scale effective gravity models, *eg.* see [32]-[35].

In the regime, the near horizon geometry for the laboratory black hole (78) may be worked out in a similar way to that for the semi-classical black hole in the previous section. The L-plane may appropriately be defined by the plane polar coordinates $(\hat{\rho}, \hat{\omega})$, where

$$\left(1 - \frac{\hat{Q}^2_{\rm eff}}{r}\right) = \frac{\hat{\rho}^2}{4\hat{Q}_{\rm eff}} \ . \tag{79}$$

Then, the near horizon limit is described by a new coordinate $\hat{\rho}$ for its small value and by $\hat{\omega}$ for $0 \le \hat{\omega} \le 2\pi$. In the limit $r \rightarrow \hat{Q}_{\rm eff}$, the laboratory black hole (77) is simplified using eq.(78). The near horizon geometry becomes

$$ds^2 = d\rho^2 + \frac{4\rho^2}{\hat{Q}^2_{\rm eff}} d\tau^2 + 8\hat{Q}^2_{\rm eff} \Big[(1 - \cos 2\phi) d\theta^2 + (1 + \cos 2\phi)\sin^2\theta d\phi^2\Big] \ . \tag{80}$$

In the regime, the relevant geometry reduces to

$$ds^2_L = d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\omega}^2 \ , \tag{81}$$

where $\hat{\omega} = 2\tau/\hat{Q}_{\rm eff}$. As a result, the Euclidean time becomes $\tau \rightarrow \tau + \pi\hat{Q}_{\rm eff}$. It implies that the radius of time like coordinate becomes small in the near horizon geometry. The solution (81) describes a two dimensional laboratory black hole in the Plankian regime.

Then the relevant two dimensional laboratory (Schwarzschild) black holes may be obtained from the eq.(78). The black hole solution is described by

$$ds^2_L = dr^2 + \left(1 - \frac{\hat{Q}^2_{\rm eff}}{r^2}\right)\left(1 + \frac{\hat{Q}^2_{\rm eff}}{r^2}\right)^{-1} d\tau^2 \ . \tag{82}$$

Since the black hole is obtained in the gravity decoupling limit $r \rightarrow \hat{Q}_{\rm eff}$, it describes the near horizon geometry (81). It suggests that the scattering of noncommutative strings [14] may be seen to be associated with the production of two dimensional laboratory black holes (82) at the interaction vertex. These black holes are presumably short lived ultra-high energy states in the noncommutative string theory. They may may be seen to describe the D-string in the theory.

Then the Hawking temperature for the laboratory black hole (77) is appropriately governed by the the eqs.(57) and (68) in the gravity decoupling limit. It is given by

$$T^{\rm Lab}_{\rm Hawking} = \frac{1}{8\pi\hat{Q}_{\rm eff}} \ . \tag{83}$$

It implies that the Hawking temperature increases as $\hat{Q}_{\rm eff}$ decreases by the emission of quantum radiation from the laboratory Schwarzschild black hole geometry (77). The process continues until all the nonlinearity, *i.e.* Θ-terms in $\hat{Q}_{\rm eff}$, are radiated out. Then $\hat{Q}_{\rm eff}$ takes a constant value $\hat{Q}$ and Hawking temperature attains its maximum *i.e.* the Hagedorn temperature in the theory.

7 Concluding remarks

To summarize, we have considered the evolution of gravity, starting from the classical to the Planckian regime on a D_3-brane in a string theory. The noncommutative set-up being natural, it has incorporated a generalized notion to the various brane geometries. In particular the self-duality condition on the gauge field, in its counter-part with ordinary geometry, allows one to incorporate all order gauge curvatures in the theory. As a result, the theory turns out to be an exact in Θ. It has allowed us to explore some aspects of black hole geometries, including that of an exact conformal field theory obtained, otherwise, in an $SL(2,R)/U(1)$ gauged Wess-Zumino-Witten model [31].

In context, we have dealt with the problem in two steps. Firstly, we have exploited the existing noncommutative scaling in the cartesian coordinates of the frame-work, to approximate the, otherwise, orthogonal spherical polar coordinates in the theory. It was shown that the radial coordinate r and the Euclidean time τ receive lower bounds, of the order of Planck scale, in the theory. There, the spherical geometry, described by the angular coordinates (θ, ϕ), is rather modified to S_2^{Θ}. Interestingly, a transformation between a spherical polar coordinate $\mathcal{J}_S$-system and its noncommutative counter-part $\tilde{\mathcal{J}}_S$ was argued. It was shown that an orthogonal rotation of (r, θ)-plane, by an arbitrary angle θ, in $\mathcal{J}_S$-coordinates leads to $\tilde{\mathcal{J}}_S$-system. In other words, the broken spherical symmetry in the noncommutative set-up may be restored by taking into account the complete configuration space of the rotation angle θ. The underlying idea was crucial to address a semi-classical solution in spherical polar coordinates in the noncommutative frame-work.

As a part of the second step, we have obtained a GS-black hole solution on a noncommutative D_3-brane in a static gauge. In the effective theory, the black hole was shown to be governed by a trivial noncommutative gauge field. On the other hand, a charged black hole underlying the notion of the GRN-black hole was obtained for a nontrivial, noncommutative gauge field in the frame-work. The black hole solutions were argued to be exact to all orders in Θ. Both, the GRN- and the GS-black holes contain higher order corrections, in $1/r$, with respect to their semi-classical cousins. Using a shift transformation, the obtained generalized solutions were re-expressed in the spherical polar coordinates. In the process, the mass and charge of the generalized black holes were shown to receive noncommutative corrections, which in turn may be viewed as a stretch in the event horizons of the GS- and GRN-black holes. Since $\Theta \neq 0$ even in the classical regime, the stretch in the event horizons was argued to be a general phenomenon in the black hole physics.

Interestingly, different black hole geometries were shown to be originated from the GRN-

black hole for various values of its effective mass and charge in the two extreme regimes. For instance, the semi-classical regime in absence of an $U(1)$ gauge charge was shown to describe a typical Schwarzschild black hole. However, the presence of a gauge charge gave rise to two independent semi-classical solutions. They are described (i) by switching-off the nonlinearity and (ii) by switching-on the nonlinearity in the Maxwell theory. In absence of nonlinearity, the GRN-black hole in the semi-classical regime was shown to govern a typical RN-black hole. However, the presence of nonlinearity in the **E**-field, a priori, did not introduce any substantial change in the nature of the RN-black hole. It had modified the parameters in the RN-black hole geometry and subsequently introduced stretches in the event horizons. In addition to the stretch, the Θ parameter was argued to decouple the underlying 4-dimensional effective space-time into two independent sectors, *i.e.* L- and $\perp$-spaces. Incorporating the appropriate noncommutative scales for a fixed Θ, the classical regime on the D_3-brane was shown to represent a two dimensional near horizon black hole geometry. At the other extreme, *i.e.* in the Planckian regime, the GRN-black hole geometry was argued to be governed exclusively by the nonlinear **E**-field in the theory. It was shown that the GRN-black hole precisely reduces to the GS-geometry in the regime, which in turn gave rise to the two dimensional laboratory black holes in the theory. Our analysis suggests that the D-string in the Planckian regime may be viewed as a two dimensional laboratory black hole. It is inspired by the one of old conjectures of 't Hooft in the context of a particle dynamics at Planck scale [24].

In addition, we have revisited the phenomenon of Hawking radiation from the GRN-black hole in the noncommutative formalism. Since different black holes were shown to be originated from the GRN-black hole, the Hawking temperature in the theory may be argued to fluctuate from its ever increasing behaviour from the classical to the Planckian regime. For instance, the temperature begins to increase with the typical Schwarzschild black hole, followed by a sharp fall to describe the RN-geometry in the classical regime and finally it attains the Hagedorn temperature in the Planckian regime. The passage from the semi-classical to the Planckian regime in the noncommutative frame-work was argued to be governed by a series of quantum radiations due to the **E**-field. The increase in nonlinear **E**-field, in each step, lead to the gradual decoupling of the (Schwarzschild) mass $M \rightarrow \hat{Q}/\sqrt{G_N}$, followed by a series of radiations in the Planckian regime. The decoupling of a series of nonlinearity in the **E**-field dominates at the Planck scale and finally the Hawking radiation ceases at $\mathbf{E}_c$.

To conclude, the closed string decoupling limit at the Planck scale is a powerful technique to explain some of the quantum gravity effects. Though Einstein's GTR is expected to break down at Planck scale, the string frame-work appears to forbid the possibility. In other words, the GTR coupled to the nonlinear Maxwell's theory on a D_3-brane provides a plausible frame-

work at Planck scale consistent with the special theory of relativity. At this point of time, perhaps the noncommutative frame-work provides an appropriate forum to address some of the open questions in quantum gravity.

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